*II. A PRELIMINARY ATTEMPT TO DEVISE A TEST OF THE ABILITY OF HIGH-SCHOOL PUPILS IN THE MENTAL MANIPULATION OF SPACE RELATIONS

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TEST A

1. A carriage is moving in the direction indicated by the arrow. Show by a curved arrow the direction in which the wheels of the carriage are turning. ←

Percentage of incorrect answers out of a total of 337—4.6; pupils appear to guess rather than think through the problem. Question is defective because of the probability of guessing the correct answer being ½; question is omitted in tests C and D.

2. A circular piece of paper is folded through the center so that the curved edges fall together. This is then folded a second time making all the edges fall together, and this is folded again in the same way. After the third folding, into how many parts is the piece of paper divided by the creases?

Percentage of incorrect answers—23.4, distributed as follows:

59% answer 6. Pupils may have thought that since folding once gives 2, folding three times will give 3×2, or 6; or they may have thought, folding once gives 2, folding a second time gives 4; hence folding a third time gives 6—by the series 2, 4, 6, etc.

The error here seems to be due to thinking in abstract terms, and without due reference to the spatial data.

19% answer 16; folding four times instead of three.

22% answer one of these: 5, 4, 12, 7, 14, 18.

3. A square is to be drawn on the blackboard and then divided into four equal small squares. What is the least

*Continued from page 605 of the October issue of the *Review*.

number of straight lines that must be drawn to represent the entire figure?.....

Percentage of incorrect answers—25.4, distributed as follows:

50% answer 2; evidently counting the inside lines only. In this case the error is due to failure to comprehend the verbal statement of the question.

15% answer 4; counting the outside lines only, or counting the segments of the two inside cross lines.

35% answer: 10, 8, 7, 1, 3, 5, 12, 16, or 20. Question is included in test C but is reworded.

4. A man starts in a car from point A, travels ten miles south to B, then ten miles west to C, then ten miles south to D. A second man goes from A to D traveling west till he is just north of D, then traveling south to D. How many miles west does the second man go?.....How many miles south does he go?.....

Percentage of incorrect answers—40.5, distributed as follows:

16% answer 10, 10; due to general impression of 10 miles.

11% answer 20, 10; answers just reversed.

10% answer 30, 20.

The remainder answer various combinations of 0, 10, 20, and 30.

Errors seem to result from guessing, or too short a span of attention.

5. What is the least number of times that a square piece of paper must be folded so that the creases will divide it into sixteen smaller squares?.....

Percentage of incorrect answers—26, distributed as follows:

50% answer 8; folding doubles and 2 times what equals 16.

15% answer 6; 3 creases each way necessary, hence 3+3 gives 6.

35% answer 5, 8, 2, 1, 7, 10, 64.

Errors result from thinking mechanically and without regard to the concrete situation presented by the problem.

6. A boatman is rowing north facing south as he rows. When he pulls on the oar in which direction does the part of the oar in the water move?.....

Percentage of incorrect answers—23.6:

87% answer north; boat goes north hence oar moves north.

13% answer west, east, no motion.

Errors result from failure to manipulate space relations.

Question is omitted in tests C and D; "south" was considered the correct answer but the answer in reality varies too much according to possible positions of the oar at any instant.

7. A boy has a square box lid on which he can lay just three rows of three marbles each. If now he places on top of this first layer of marbles a second layer in which each marble touches four marbles of the first layer, how many marbles are there in the second layer?

Percentage of incorrect answers—40.5:

23% answer 6; unable to visualize entire first layer.

18% answer .2; see next answer.

9% answer 2 and ¼; purely mechanical solution, 9 divided by 4 gives 2¼.

50% answer 12 (4×3), 3, 1, 9, 13, 16, 5, 7, 8, 10, 27, 36 (9×4).

8. It is 6:48 o'clock. Suppose the hands change places so that the large hand takes the place of the small hand while the small hand takes the place of the large hand; what time would it then be?.....

Percentage of incorrect answers—33.2 (9:30 to 9:35 being counted correct):

18% answer 10:30; 6% answer 8:30; 4% answer 9:06; showing inability to fix and hold one hand in place while locating the other hand.

Remainder give some twenty different answers, most interesting group of which is 48:6 (answer of 2%), 4:86 (answer of 1%), 4:6 (answer of 2%), 4:30 (answer of 1%), the first of which is an excellent example of a purely mechanical reaction to the verbal symbol "change places."

9. Two sticks of equal length are fastened together at the end by a kind of hinge that bends in any direction. Each stick is then suspended at the middle by a string. If the free end of one stick is pulled down a little what does the free end of the other do?.....

Percentage of incorrect answers—69.5:

64¾% answer "it moves up"; failure to visualize entire situation, attention being limited to first stick.

20% answer "it remains still"; attention includes second stick but its exact relation to first is not noted.

Question is omitted in tests C and D. Too many failures seeming to to show that the problem was not clear to the pupils.

10. Three men, A, B, and C, start from the same point, A going north three miles, B going east three miles, and C going west three miles. If the straight line distance between them is measured which men are farthest apart?..... Which men are the same distance apart?.....

Percentage of incorrect answers—45.

Incorrect answers result either from inability to visualize or otherwise hold the three men in their final positions or from failure to understand what is meant by the "straight line distance between them."

11. The following words are to be printed in capital letters made up of straight lines only. For example: man = MAN. Without printing the words state the least number of straight lines that must be drawn to print each of the following: knit..... have..... meat..... while.....

Percentage of incorrect answers: "knit"—16.8; "have"—13.4; "meat"—17.6; "while"—34.8.

Question is omitted in texts C and D. Pupils with very low scores too frequently score high on this question; difficult to prevent pupils from using finger or pencil to draw the word without actually making the lines, and counting the strokes.

12. The lock on the door of my room is so made that when I am inside and wish to unlock it I must turn the key in the direction opposite to that in which the hands of the clock move. When I am outside, in which direction must I turn the key to lock it? Show by a curved arrow:.....

Percentage of incorrect answers—69.

Question is defective because of the probability of guessing the correct answer being ½; question is omitted in tests C and D.

13. I have a small checkerboard that is divided into twenty-five small squares only. If the rules are that I can move my men from right to left, left to right, or up, only, how many

moves will it take to get a man from the square in the lower right hand corner to the center square? One move means from one square to the next square on the right, on the left, or above.....

Percentage of incorrect answers—75.2:

22% answer 3; goes diagonally and counts number of squares.

21% answer 5; counts number of squares instead of number of moves.

11% answer 2; counts moves but goes diagonally.

10% answer 6; counts number of squares to left, 1, 2, 3, then number of squares up 4, 5, 6.

Other errors show that some thought the board consisted of 25 squares on a side. Incorrect answers are due to hasty or defective reading resulting in failure to get the exact conditions of the problem.

Question is omitted in tests C and D—question is primarily a "directions test."

14. How many lines must be drawn on the blackboard to represent a box if each edge of the box is represented by one line?.....

Percentage of incorrect answers—45.7:

35% answer 9; counting visible edges only.

13% answer 8.

10% answer 4; using rectangle to represent a box.

Each of these errors due to failure to note that "each edge of the box is to be represented by one line."

Remaining answers are 6, 7, 11, 16, 13, 12, 14, 20.

Question is reworded in test D in an effort to call attention more directly to fact that each edge is to be represented by a line.

15. A block of wood in the shape of a cube is painted on all sides. It is then cut into eight small cubes. How many of these small cubes are painted on four sides only?..... How many are painted on three sides only?..... How many are painted on two sides only?.....

Percentage of incorrect answers—67.3:

15¾% answer 0, 4, 4; 6% answer 4, 4, 0; 5% answer 4, 4, 2; remaining answers represent over 20 different combinations of numbers from 1 to 32. Difficulty seems to be the inability to visualize. Thus in the first answer the

four cubes on the front face can be seen to be painted on three sides while the remaining cubes are hazy and hence, largely by guess are put down as painted on two sides.

16. Five points are placed on the blackboard as if they represented the corners of a five-sided figure. How many different straight lines may be drawn between these points?

. . . . . .

Percentage of incorrect answers—83.5:

35% answer 5; outside lines only between the points; hasty conclusion on the part of the pupil as to what the question calls for precludes the other possibilities involved in the question.

12% answer 9; outside lines plus the lines from one point to each of the other four.

Remaining fifteen answers range from 1 to 25.

Question is not used in test C or D—occasionally the poor pupils get the correct answer by multiplying five by two, as appeared on inquiry of the same.

17. I have a square piece of paper on the desk in front of me. I fold it once by placing the bottom edge on the top edge and creasing. I then fold over the left side of this, placing the left edge on the right edge and creasing. I next cut a small piece out of the middle of the folded edge on my left Make a drawing of the paper as it would appear to me when unfolded, showing the creases and what results from the cutting:

Percentage of incorrect answers—77.2.

56% drew the intersecting lines representing the creases correctly but placed the piece cut out at the intersection of these lines. This and the large variety of remaining errors seem to result from inability to manipulate the spatial data involved.

18. I have a square piece of paper on the desk in front of me. I fold it once placing the upper left hand corner on the lower right hand corner and creasing. I then fold this, placing the lower left hand and upper right hand corners side by side on the lower right hand corner and creasing. I now fold this by placing the upper right hand corner of it on the lower left

hand corner and creasing. Make a drawing of the paper as it would appear to me when unfolded showing the creases:

Percentage of incorrect answers—60.

Practically all incorrect answers have the first diagonal line drawn correctly but after this is drawn the answers take all possible forms of guessing.

19. Draw this figure as it would appear from behind to the teacher if he were holding it before you:

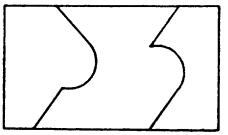

Figure 1

Percentage of incorrect answers—49.

Question is defective—some pupils having studied symmetry in first or second year mathematics had this application of it and therefore found the question relatively easy, while others of equal ability but without the principle of symmetry were unable to do it.

20. Each large square below is to be filled by placing the small figures to the right in the large square. Without

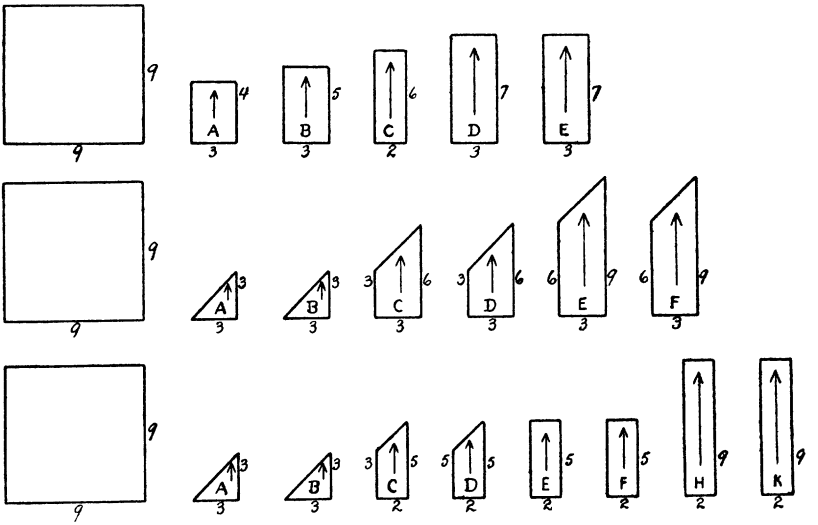

Figure 2

drawing any lines, show how you would do this by writing in the letters with the arrow to show the position of each small figure when so placed in the large square. When they are placed correctly all the space in the large square will be filled up. The numbers show the size. Remember to draw no lines. Write letters and arrows only:

Percentage of incorrect answers: first set—27.8; second set—37; third set—72. Failure seemed to be due to inability to hold in mind the sections already placed while locating remaining sections or to inability to see the vacant section remaining in the square and its relation to the section or sections still to be placed.

Question is rewarded in test D in an effort to simplify directions as well as to prevent pupils from drawing in the smaller figures on the square, which a number did in this test despite the directions.

TEST B

1. A boy has three wheels placed in a row with rims touching so that when he turns one wheel each of the others turns. He now turns the first wheel in the direction opposite to that in which the hands of a clock turn. Show by a curved arrow the direction in which the third wheel turns.....

Percentage of incorrect answers out of a total of 352—15.6.

Question is defective and is not included in test C or D, the probability of guessing the correct answer being ½.

2. A boy is ringing a bell by moving it in front of him from right to left and left to right. In which direction is the bell moved when the clapper hits the side of the bell on the boy's right?.....

Percentage of incorrect answers—16.2.

Question is not included in C or D—same reason as No. 1.

3. If you take a sheet of paper, fold it once making the bottom edge fall on the top, fold it a second time making the folded edge fall on the top, then fold it a third time in the same way, how many layers of paper have you after the last folding?.....

Percentage of incorrect answers—40.3, distributed as follows:
67% answer 6; same error as that noted in test A No. 2.
17% answer 4; 2, plus second, plus third folding, equals four.
Remaining answers: 3, 16, 1, 2, 12, 5, 9, 14.

4. A man starts in a car from point A, travels ten miles east to B, then ten miles north to C, then ten miles east to D. A second man goes from A to D traveling north till he is just west of D, then traveling east to D. How many miles north does the second man go?..... How many miles east does he go?.....

Same as test A No. 4.

5. A square cake is cut by drawing a knife from each corner to the opposite corner and from the middle of each side to the middle of the opposite side. How many strokes does the knife make?..... Into how many pieces is the cake cut?.....

Percentage of incorrect answers—39.6:
21% answers 2, 4; counting strokes from one corner and one side only; the pupil, satisfied with the first answer that comes to mind neglects other factors or possibilities in the situation.
16% answer 4, 12; may count lines instead of pieces.
14% answer 3, 6; same error as first answer.
Remaining answers are various combinations of 2, 4, 5, 6, 7, and 8.

6. A river runs due south. A man starts to row across from the west side and as he rows always keeps the boat pointing east. If the river runs just as fast as the man can row in still water, in which direction does the boat travel as it crosses the river?.....

Percentage of incorrect answers—49.3:
36% answer south; river flows south therefore boat goes south.
35% answer east; boat points east therefore it travels east.
In both answers above, the totality of factors is neglected in favor of a single factor.
Remaining answers are: north, southwest, still, northeast, west.

Question is not included in test C or D since it involves a principle of physics which first year pupils might not be expected to recognize as presented in this question.

7. Four points are placed on the blackboard as if they represent the corners of a four-sided figure. How many different lines can be drawn between these points?.....

Percentage of incorrect answers—26:

38% answer 4; outside lines only.

23% answer 2; inside lines only.

11% answer 8.

8% answer 3; lines from one point to each of the other three.

Other answers: 1, 12, 16.

Mistakes seem due primarily to failure to perceive all the elements or factors in the situation.

8. It is 5:12 o'clock. Suppose the hands change places so that the large hand takes the place of the small hand while the small hand takes the place of the large hand; what time would it then be?.....

Percentage of incorrect answers—29.6 (2:25 to 2:27 being counted correct):

25% answer 12:5 or 12:05; compare this with the number answering 48:6 in test A No. 8; illustrating again a purely mechanical reaction to the verbal symbol "change places."

Other answers illustrate the same facts already noted in test A, No. 8.

9. Five pennies are placed side by side in a row on the table. A second row is placed above this so that each penny touches two pennies in the first row. How many pennies are there in the second row?..... How many pennies in a third row placed in the same way as the second row?..... Altogether how many pennies could be so placed above the first row of five pennies, not counting the first row?.....

Percentage failing to answer first two parts correctly—31:

35% answer 3, 2, 6; 17% answer 3, 2, 5; span of attention not sufficient to take in the five marbles but has no difficulty with the three.

3.5% answer 10, 20, 30; thought reaction mechanical; $2\times5=10$, $2\times10=20$.

Remaining answers include 21 combinations of numbers from 1 to 30.

In test C the third part of the question is omitted as a large number either counted the first row or failed to count the pennies above the third row, though answering the first two parts correctly.

10. Three men, A, B, and C, start from the same point, A going west five miles, B going south five miles, and C going north five miles. If the straight line distance between them is measured which men are farthest apart?..... Which men are the same distance apart?.....

Results similar to those for 10 in Test A.

11. The following words are to be printed in capital letters made up of straight lines only. For example: man = MAN. Without printing the words state the least number of straight lines that must be drawn to print each of the following: hint..... melt..... wave..... knife.....

Results similar to those for 11 in Test A.

12. A small box is held in place in the center of a large box by rods, one rod being stretched from each corner of the small box to the corresponding corner of the large box. How many rods are there?.....

Percentage of incorrect answers—60.7:

82% answer 4; counting rods from four corners on top of box only, or considering corner to mean the diedral angle.

Other answers: 2, 1, 6, 3, 5, 16.

Question is omitted in C and D; it is ambiguous, since by "corner" one does not know whether the diedral angle or triedral angle is meant.

13. A child has nine small sticks of equal length. He tries to see how many three-sided figures he can make by placing the sticks in any arrangement on the floor. What is the largest number he can make by placing the figures next to each other?.....

Percentage of incorrect answers—49.3:

44% answer 3; either fail to place figures together or divide 9 by 3.

6% answer one of these: 111111111, 444, 99999999; by "figure" the pupil understands the question to mean an arithmetical number, or the thought last in mind "what is the largest number" alone decides the answer shutting out all other conditions of the problem.

Remaining answers are: 5, 9, 1, 8, 2, 6, 54.

Question is omitted in tests C and D; the problem intended is not readily stated in clear simple language.

14. A square is to be drawn on the blackboard and then divided into nine equal small squares. What is the least number of straight lines that must be drawn to represent the entire figure?.....

Percentage of incorrect answers—44.3:

27.5% answer 4; inside lines only.

17% answer 6; problem requires dividing base and height each into three parts, hence 3+3=6.

Other answers: 10, 9, 18, 20, 24, 5, 7, 22, 36, 17, 81 (square 9), 8, 97.

Errors seem largely due to failure to comprehend the verbal statement of the problem.

Question is reworded in test C in an effort to make the statement clearer.

15. A carpenter has a block of wood that is a perfect cube. How many times must he saw through the entire block to divide it into eight equal small cubes?.....

Percentage of incorrect answers—65.7:

22% answer 4; 12% answer 6; remaining answers are 2, 8, 7, 5, 16, 9, 1, 21, 41, 48, 256, 504. Let him who reads try to explain these impossible answers, given by first and second year high school pupils.

16. A boy winds his string about his top in the direction indicated by the arrow: ← In which direction will the top spin when thrown on the ground? Indicate by a curved arrow:

.....

Percentage of incorrect answers—79.1; many of these answers are due probably to the reasoning that since the string is unwound to spin the top, the direction of spinning is opposite to the direction in which the string is wound.

Question is omitted in C and D—the probability of guessing the correct answer being ½.

17. I have a square piece of paper on the desk in front of me. I fold it once placing the lower left hand corner on the upper right hand corner and creasing. I then fold this, placing the lower right hand corner on the upper left hand corner and creasing. I next cut a small piece out of the middle of the folded edge on my right. Make a drawing of the paper as it would appear to me when unfolded, showing the creases and what results from the cutting:

The findings on questions 17, 18, 19, do not vary from those previously noted under the corresponding questions in Test A.

18. I have a square piece of paper on the desk in front of me. I fold over the top half making the top edge fall on the bottom edge and creasing. I next fold up the lower left hand and lower right hand corners of this making them fall side by side along the folded edge. I now fold over the left half of this making it fall on the right half, edge to edge. Make a drawing of the paper as it would appear to me when unfolded, showing the creases:

19. Draw this figure as it would appear from behind to the teacher if he were holding it up before you:

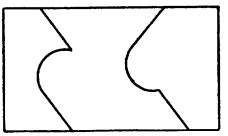

Figure 3

20. Each large square below is to be filled by placing the small figures to the right in the large square. Without drawing any lines, show how you would do this by writing in the letters with the arrow to show the position of each small figure when so placed in the large square. When they are placed correctly all the space in the large square will be filled up. The numbers show the size. Remember to draw no lines. Write letters and arrows only:

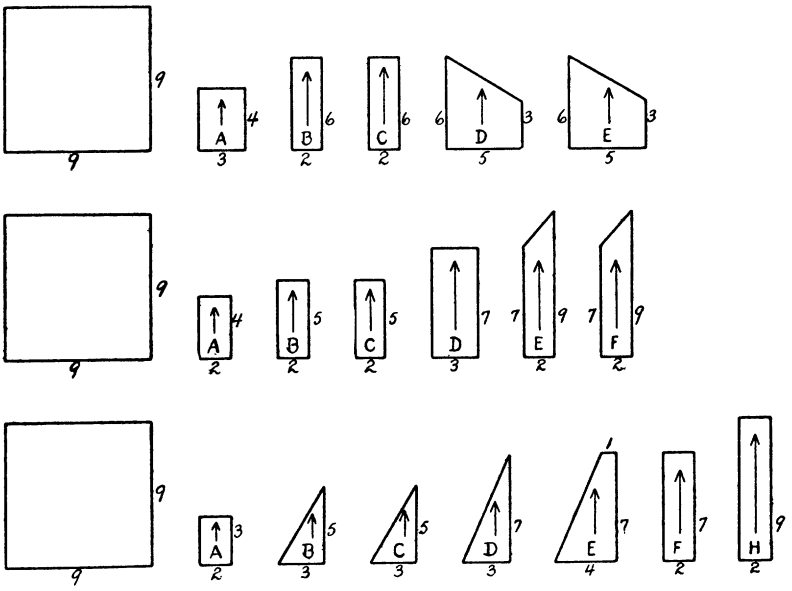

Figure 4

Of the six sets of figures used in this question and in Test A, No. 20, only three are used in Tests C and D, it being found that some pupils develop a general method of attack after working about three such problems thereby lessening to some extent the value of the exercise as a test of ability to grasp space relations.

Without permitting ourselves an excursion into the significance for classroom procedure of some of the glaring errors noted in these answers to problems that are not, for the most part, either fantastical or beyond the realm of the pupil's everyday experience, we shall conclude this part of the study with a summary list of the principal types of errors that may be observed in the various answers.

TYPES OF ERRORS

1. Guessing; snap judgments or hasty conclusions resulting from desire to economize time and effort or from a lazy unwillingness to carry through the various mental steps necessary to obtain the final correct result.

2. Failure to comprehend the verbal statement of the problem; hasty or defective reading leads to a wrong interpretation as to what the problem calls for; the reader skims over the first part of the question in his desire to find out what is called for at the close and so misses fundamental conditions in the problem.

3. Failure to make association between the verbal statement and the concrete situation presented by the problem; pupil's answer is a result of certain thought processes mechanically induced and carried to a conclusion without any or due reference to the actual situation.

4. Thinking primarily or solely in abstract terms and without any or due reference to the spatial data or relations involved in the situation; e.g., substitution of arithmetical computation for visual or other imagery.

5. Totality of factors in a given situation neglected in favor of single idea in thought suggested by some phrase in question or factor in situation which is sufficiently vivid or insistent to prevent pupil from laying hold of entire situation; e.g., idea last uppermost in mind is sole determinant of answer to the question.

6. Span of attention too short resulting in:

a) Inability to hold all the elements of a situation before one.

b) Failure to see each of the significant elements in the situation.

c) Inability to hold before the mind the old elements of a situation while adding new elements.

7. Inability of the "constructive imagination" to manipulate the concrete visual data or space relations by means of visual, verbal, tactual or other imagery.

While the types of errors in the preceding summary are listed roughly in the order of the frequency of their occurrence the most frequent being the last mentioned type, definite

measures of such frequency are not presented because the data are not always at hand for classifying each incorrect answer as distinctly of one type rather than another. For the same reason it is not yet possible to make a study of individual test papers to note the frequency of occurrence of these errors as a means of diagnosing the specific abilities or inabilities of each individual. Nor again can the errors above listed, the elimination of which through the development of certain specific abilities is one of the avowed aims of a course in geometrical mathematics, be as yet made the basis of any statement regarding the extent to which such courses develop, for example, the ability to deal with space relations, or the ability to lay hold of each of the concrete elements in a problematic situation, or the ability to comprehend fully the verbal statement of a problem. These, and a number of related questions must await further experimentation and study along this line.

We may now turn in conclusion to the two final tests that have been constructed on the basis of the material thus far provided by our study. Two sets of questions have been formulated in order that one may be used as a pre-training test with the other as the post-training test. The questions comprising one set are entirely different from those in the other, so that the practice effect resulting from the pre-training test should be reduced to a minimum.

It will also be noted in these proposed tests that an attempt has been made to arrange the questions in the form of a cycle of ten, the questions in each cycle being arranged roughly in order of difficulty. By this arrangement it is hoped to secure more definite check upon the nature of the error occurring in any answer; whether, for example, it is the result of random guessing, or of failure to comprehend the verbal statement, or of inability to carry through the necessary readjustment of the spatial data involved, etc. Each test is limited to three cycles because it has been found that some high school pupils are

quick to seek some general mode of attack as an aid in solving any particular type of problem, with the result that a longer series than three might become in part a test of the pupil's ability to generalize from particular cases.

Finally it should be understood that these new sets of questions are not presented as in any way perfected class tests of ability to deal with space relations. Instead they are given simply as indicating the results to date of the study. Further experimentation, which it is hoped to carry on this coming year, may materially alter both the results and the mode of procedure herein reported. In the belief that other teachers, particularly those in drawing and geometrical mathematics, are equally interested in the problem we are here working on it has been deemed desirable to give in full the "Directions for Giving the Test," "Directions to Pupils," and both test C and test D.